Option Pricing Via The Binomial Tree

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In this white paper we will price a call option using a Binomial Tree. We will solve the following hypothetical problem...

Our Hypothetical Problem

Assume that we are tasked with calculating the value of a call option on ABC Company stock given the following model parameters...

Table 1: Model Parameters

Symbol	Description	Value
S_0	Stock price at time zero	\$80.00
X_T	Call option exercise price at time T	\$70.00
T	Option term in years	1.00
n	Number of time periods in [0,T]	2
α	Risk-free rate (percent)	4.00
σ	Stock annual log return volatility (percent)	35.00

Question: What is the no-arbitrage price of the call option at time zero?

Stock Price Equations

We will define the variable T to be option term (in years), the variable t to be the length of one time period (in years), and the variable n to be the total number of time periods (an integer) in the time inteval $[0, T]$. Using the information in Table 1 above the equation for the length of one time period is...

$$t = \frac{T}{n} = \frac{1.00}{2} = 0.50 \tag{1}$$

The log of stock price can either increase or decrease by a fixed amount over the time interval $(n-1)\,t, n\,t]$. Using Equation (1) above and the information in Table 1 above the equation for stock price at time $n\,t$ as a function of stock price at time $(n-1)\,t$ is...

$$S_{n\,t} = S_{(n-1)\,t}\,\mathrm{Exp}\left\{\sigma\sqrt{t}\right\} \text{ ...or... } S_{n\,t} = S_{(n-1)\,t}\,\mathrm{Exp}\left\{-\sigma\sqrt{t}\right\} \tag{2}$$

We will define the variable q to be the risk-neutral probability that the stock will move up over the time interval $[(n-1)\,t, n\,t]$. Using Equation (2) above the equation for expected stock price at time $n\,t$ under the risk-neutral Measure Q is...

$$\mathbb{E}^Q\left[S_{n\,t}\right] = q\,S_{(n-1)\,t}\,\mathrm{Exp}\left\{\sigma\sqrt{t}\right\} + (1-q)\,S_{(n-1)\,t}\,\mathrm{Exp}\left\{-\sigma\sqrt{t}\right\} \tag{3}$$

Since all assets earn the risk-free rate under the risk-neutral probability measure we can also define expected stock price at time $n\,t$ under the risk-neutral Measure Q to be...

$$\mathbb{E}^Q\left[S_{n\,t}\right] = S_{(n-1)\,t}\,\mathrm{Exp}\left\{\alpha\,t\right\} \tag{4}$$

If we equate Equations (3) and (4) above and solve for q then the equation for the risk-neutral probability q is...

$$q = \left(\text{Exp}\left\{\alpha\, t\right\} - \text{Exp}\left\{-\sigma\sqrt{t}\right\}\right) \Big/ \left(\text{Exp}\left\{\sigma\sqrt{t}\right\} - \text{Exp}\left\{-\sigma\sqrt{t}\right\}\right) \tag{5}$$

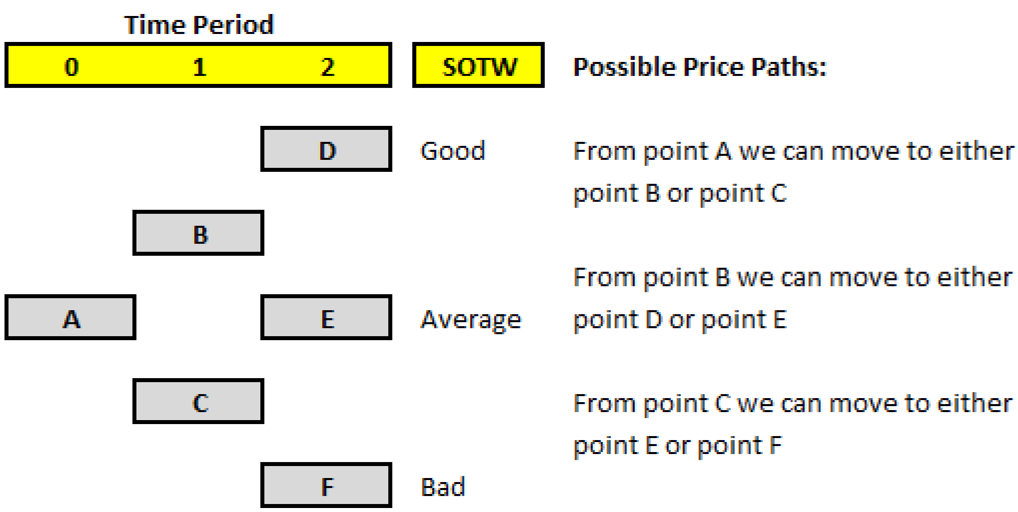
Using Equation (5) above and the information in Table 1 above the value of risk-neutral probability q is...

$$q = \left(\text{Exp}\left\{0.04 \times 0.50\right\} - \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\}\right) \Big/ \left(\text{Exp}\left\{0.35 \times \sqrt{0.50}\right\} - \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\}\right) = 0.4788 \tag{6}$$

Using the Binomial distribution the probablity of k up moves and $n-k$ down moves over the time inteval $[0, T]$ is...

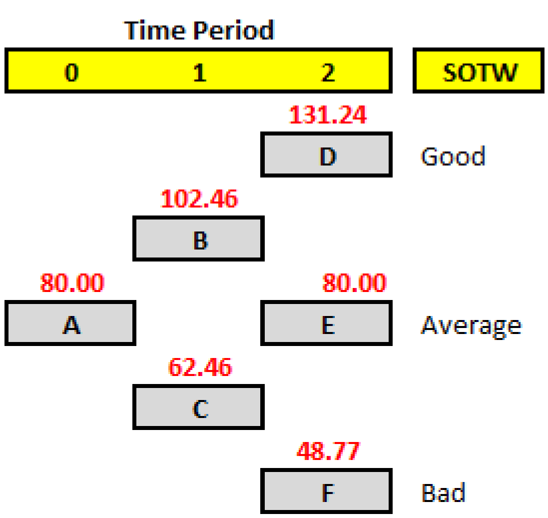
$$\text{Prob}\left[(k)\text{ up moves and }(n-k)\text{ down moves}\right] = \frac{n!}{k!(n-k)!}\, q^k\, (1-q)^{n-k} \tag{7}$$

If we divide the time interval $[0, T]$ into $n = 2$ time periods then there are $n + 1 = 3$ possible states of the world (SOTW) at time T. We will model asset price paths via the following binomial tree...



Stock Price Paths Via Forward Recursion

Using Equation (2) above the stock's price paths via forward recursion contingent on whether there is one up move or one down move in each discrete time period is...



Using the information in Table 1 above the equation for stock price at time zero (point **A**) is...

$$A = S_0 \text{ ..such that... } A = 80.00 \tag{8}$$

If we are currently at point **A** and there is an up move in the stock price then we move from point **A** to point **B** in the tree above. Using the information in Table 1 above the equation for stock price at point **B** is...

$$B = S_0 \,\text{Exp}\left\{\sigma\sqrt{t}\right\} \text{ ..such that... } B = 80.00 \times \text{Exp}\left\{0.35 \times \sqrt{0.50}\right\} = 102.46 \tag{9}$$

If we are currently at point **A** and there is a down move in the stock price then we move from point **A** to point **C** in the tree above. Using the information in Table 1 above the equation for stock price at point **C** is...

$$C = S_0 \,\text{Exp}\left\{-\sigma\sqrt{t}\right\} \text{ ..such that... } C = 80.00 \times \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\} = 62.46 \tag{10}$$

If we are currently at point **B** and there is an up move in the stock price then we move from point **B** to point **D** in the tree above. Using Equation (9) above and the information in Table 1 above the equation for stock price at point **D** is...

$$D = B \,\text{Exp}\left\{\sigma\sqrt{t}\right\} \text{ ..such that... } D = 80.00 \times \text{Exp}\left\{0.35 \times \sqrt{0.50}\right\} \times \text{Exp}\left\{0.35 \times \sqrt{0.50}\right\} = 131.24 \tag{11}$$

If we are currently at point **B** and there is a down move in the stock price then we move from point **B** to point **E** in the tree above. Using Equation (9) above and the information in Table 1 above the equation for stock price at point **E** is...

$$E = B \,\text{Exp}\left\{-\sigma\sqrt{t}\right\} \text{ ..such that... } D = 80.00 \times \text{Exp}\left\{0.35 \times \sqrt{0.50}\right\} \times \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\} = 80.00 \tag{12}$$

If we are currently at point **C** and there is an up move in the stock price then we move from point **C** to point **E** in the tree above. Using Equation (10) above and the information in Table 1 above the equation for stock price at point **E** is...

$$E = C \,\text{Exp}\left\{\sigma\sqrt{t}\right\} \text{ ..such that... } D = 80.00 \times \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\} \times \text{Exp}\left\{0.35 \times \sqrt{0.50}\right\} = 80.00 \tag{13}$$

If we are currently at point **C** and there is a down move in the stock price then we move from point **C** to point **F** in the tree above. Using Equation (10) above and the information in Table 1 above the equation for stock price at point **F** is...

$$F = C \,\text{Exp}\left\{-\sigma\sqrt{t}\right\} \text{ ..such that... } F = 80.00 \times \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\} \times \text{Exp}\left\{-0.35 \times \sqrt{0.50}\right\} = 48.77 \tag{14}$$

Stock Price Paths Via The Binomial Distribution

Note that using Equation (2) above we can write the equation for stock price at time T as follows...

$$S_T = S_0 \,\text{Exp}\left\{\Big(\text{number of up moves} - \text{number of down moves}\Big)\sigma\sqrt{t}\right\} \tag{15}$$

We will define the variable k to be the number of up moves and $n-k$ to be the number of down moves. Using this new variable and the information in Table 1 above we can rewrite Equation (15) above as...

$$S_T = S_0 \,\text{Exp}\left\{k\,\sigma\sqrt{t} - \Big(n-k\Big)\,\sigma\sqrt{t}\right\} = S_0 \,\text{Exp}\left\{\Big(2\,k-n\Big)\sigma\sqrt{t}\right\} \text{ ...where... } 0 \le k \le n \tag{16}$$

Using Equation (16) above we can rewrite Equations (11), (12), (13) and (14) above as...

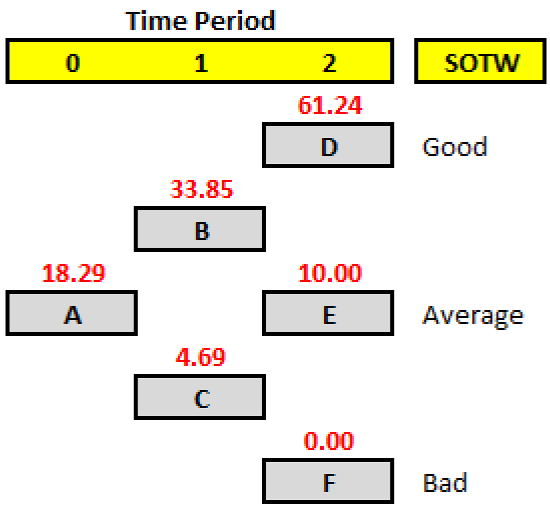
$$D = S_0 \,\text{Exp}\left\{\Big(2\,k-n\Big)\sigma\sqrt{t}\right\} = 131.24 \text{ ...where... } k=2$$
$$E = S_0 \,\text{Exp}\left\{\Big(2\,k-n\Big)\sigma\sqrt{t}\right\} = 80.00 \text{ ...where... } k=1$$
$$F = S_0 \,\text{Exp}\left\{\Big(2\,k-n\Big)\sigma\sqrt{t}\right\} = 48.77 \text{ ...where... } k=0 \tag{17}$$

Option Price Paths Via Backward Recursion

We will define the variable C_T to be call option value at time T. Using the information in Table 1 above the equation for option value at time T is...

$$C_T = \text{Max}\left[S_T - X_T, 0\right] \tag{18}$$

Using the stock's price paths above we can determine option values at the end of each time period via backward recursion. Since there are $n+1$ states of the world at time T there will be $n+1$ possible option values at time T. The option value price paths in each discrete time period are...



If we are currently at point **D** then using Equations (11) and (18) above and the information in Table 1 above the equation for option price at point **D** is...

$$D = \text{Max}\left[131.24 - 70.00, 0\right] = 61.24 \tag{19}$$

If we are currently at point **E** then using Equations (12), (13) and (18) above and the information in Table 1 above the equation for option price at point **E** is...

$$E = \text{Max}\left[80.00 - 70.00, 0\right] = 10.00 \tag{20}$$

If we are currently at point **F** then using Equations (14) and (18) above and the information in Table 1 above the equation for option price at point **F** is...

$$F = \text{Max}\left[48.77 - 70.00, 0\right] = 0.00 \tag{21}$$

If we are currently at point **B** then using Equations (18), (19) and (20) above and the information in Table 1 above the equation for option price at point **B** is...

$$\begin{aligned} B &= \text{Exp}\left\{-\alpha\, t\right\}\left(q\, D + (1-q)\, E\right) \\ &= \text{Exp}\left\{-0.04 \times 0.50\right\}\left(0.4788 \times 61.24 + 0.5212 \times 10.00\right) \\ &= 33.85 \end{aligned} \tag{22}$$

If we are currently at point **C** then using Equations (18), (20) and (21) above and the information in Table 1 above the equation for option price at point **C** is...

$$
\begin{aligned}
C &= \mathrm{Exp}\left\{-\alpha\,t\right\}\Big(q\,E + (1-q)\,F\Big)\\
&= \mathrm{Exp}\left\{-0.04\times 0.50\right\}\Big(0.4788\times 10.00 + 0.5212\times 0.00\Big)\\
&= 4.69
\end{aligned}
\tag{23}
$$

If we are currently at point **A** then using Equations (18), (22) and (23) above and the information in Table 1 above the equation for option price at point **A** is...

$$
\begin{aligned}
A &= \mathrm{Exp}\left\{-\alpha\,t\right\}\Big(q\,B + (1-q)\,C\Big)\\
&= \mathrm{Exp}\left\{-0.04\times 0.50\right\}\Big(0.4788\times 33.85 + 0.5212\times 4.69\Big)\\
&= 18.29
\end{aligned}
\tag{24}
$$

Option Price Paths Via The Binomial Distribution

Using Equations (19), (20), (21), (22) and (23) above the equation for option price at time zero (point A) can be written as...

$$
\begin{aligned}
C_0 &= A\\
&= \mathrm{Exp}\left\{-\alpha\,t\right\}\Big(q\,B + (1-q)\,C\Big)\\
&= \mathrm{Exp}\left\{-\alpha\,t\right\}\Big(q\,\mathrm{Exp}\left\{-\alpha\,t\right\}\Big(q\,D + (1-q)\,E\Big) + (1-q)\,\mathrm{Exp}\left\{-\alpha\,t\right\}\Big(q\,E + (1-q)\,F\Big)\Big)\\
&= \mathrm{Exp}\left\{-2\,\alpha\,t\right\}\Big(q^2 D + q\,(1-q)\,E + q\,(1-q)\,E + (1-q)^2 F\Big)\\
&= \mathrm{Exp}\left\{-\alpha\,T\right\}\Big(q^2 D + 2\,q\,(1-q)\,E + (1-q)^2 F\Big)
\end{aligned}
\tag{25}
$$

Using Equations (7) and (16) above we can rewrite option price Equation (25) above as...

$$
C_0 = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\,q^k\,(1-q)^{n-k}\,\mathrm{Max}\left[S_0\,\mathrm{Exp}\left\{(2\,k-n)\,\sigma\,\sqrt{t}\right\} - X_T\,,0\right]\mathrm{Exp}\left\{-\alpha\,T\right\}
\tag{26}
$$

Using Equation (26) above and the information in Table 1 above call option value at time zero is...

Table 2: Call Option Value

Up Move	Path	Stock	Call Option Value		
k	Probability	Price	Nominal	PV	Expected
2	0.2293	131.24	61.24	58.84	13.49
1	0.4991	80.00	10.00	9.61	4.80
0	0.2716	48.77	0.00	0.00	0.00
Total	1.0000	–	–	–	18.29

The value of the call option at time zero via the Binomial distribution is $18.29.